The line element of our Universe is:

$$ds^2 = -dt^2 + a^2(t)[dx^2 + dy^2 + dz^2] \tag{1}$$

With this choice of metric we can express the Ricci scalar $R(t)$ in terms of the scale factor $a(t)$:

$$R(t) = 6\left[\frac{\ddot{a}(t)}{a(t)} + \left(\frac{\dot{a}(t)}{a(t)}\right)^2\right] \tag{2}$$

We consider the following action:

$$S = \int d^4x\sqrt{-g}\left\{\frac{{M_p}^2 R}{2} + R\bar{F}_1(\Box)R\right\} \tag{3}$$

The principle of stationary action requires:

$$\delta S = 0 \tag{4}$$

$$P^{\alpha\beta} = -\frac{2}{\sqrt{-g}}\frac{\delta S}{\delta g_{\alpha\beta}} \tag{5}$$

$$P^{\alpha\beta} = 0 \tag{6}$$

This means from paper of Conroy, Koivisto, Mazumdar, Teimouri:

$${M_p}^2 G_{\alpha\beta} + 2G_{\alpha\beta}\bar{F}_1(\Box)R + \frac{1}{2}g_{\alpha\beta}R\bar{F}_1(\Box)R - 2(\nabla_\alpha\nabla_\beta - g_{\alpha\beta}\Box)\bar{F}_1(\Box)R \tag{7}$$

$$+\sum_{n=1}^{\infty} f_{1-n}\sum_{l=0}^{n-1}\nabla_\beta R^{(-l-1)}\nabla_\alpha R^{(-n+l)} \tag{8}$$

$$-\frac{1}{2}g_{\alpha\beta}\left\{\sum_{n=1}^{\infty} f_{1-n}\sum_{l=0}^{n-1}\nabla_\sigma R^{(-l-1)}\nabla^\sigma R^{(-n+l)} + \sum_{n=1}^{\infty} f_{1-n}\sum_{l=0}^{n-1} R^{(-l+1)}R^{(-n+l+1)}\right\} = 0 \tag{9}$$

We can obtain an equivalent and more compact equation by multiplying both sides with $g^{\mu\nu}$. The trace equation is:

$${M_p}^2\left\{R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu}\right\}g^{\mu\nu} + 2\left\{R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu}\right\}g^{\mu\nu}\bar{F}_1(\Box)R \tag{10}$$

$$\frac{1}{2}g_{\mu\nu}g^{\mu\nu}R\bar{F}_1(\Box)R - g^{\mu\nu}2(\nabla_\mu\nabla_\nu - g_{\mu\nu}\Box)\bar{F}_1(\Box)R \tag{11}$$

$$+\sum_{n=1}^{\infty} f_{1-n}\sum_{l=0}^{n-1}\nabla_\mu g^{\mu\nu}R^{(-l-1)}\nabla_\nu R^{(-n+l)} - \frac{1}{2}g_{\mu\nu}g^{\mu\nu}[\sum_{n=1}^{\infty} f_{1-n}\sum_{l=0}^{n-1}\nabla_\sigma R^{(-l-1)}\nabla^\sigma R^{(-n+l)} \tag{12}$$

$$+\sum_{n=1}^{\infty} f_{1-n}\sum_{l=0}^{n-1} R^{(-l-1)}R^{(-n+l+1)}] = 0 \tag{13}$$

or:

$$-{M_p}^2R - 2R\bar{F}_1(\Box)R + 2R\bar{F}_1(\Box)R - 2(\Box - 4\Box)\bar{F}_1(\Box)R + \sum_{n=1}^{\infty} f_{1-n} \sum_{l=0}^{n-1} \nabla^{\mu} R^{(-l-1)} \nabla_{\mu} R^{(-n+l)} \quad (14)$$

$$-2\left\{\sum_{n=1}^{\infty} f_{1-n} \sum_{l=0}^{n-1} \nabla_{\sigma} R^{(-l-1)} \nabla^{\sigma} R^{(-n+l)} + \sum_{n=1}^{\infty} f_{1-n} \sum_{l=0}^{n-1} R^{(-l-1)} R^{(-n+l+1)}\right\} = 0 \quad (15)$$

or:

$$-{M_p}^2R + 6\Box\left\{\bar{F}_1(\Box)R\right\} - \sum_{n=1}^{\infty} f_{1-n} \sum_{l=0}^{n-1} \left\{\nabla_{\mu} R^{(-l-1)} \nabla^{\mu} R^{(-n+l)} + 2R^{(-l-1)} R^{(-n+l+1)}\right\} = 0 \quad (16)$$

$$-{M_p}^2R + 6\Box\left\{\bar{F}_1(\Box)R\right\} - \sum_{n=1}^{\infty} f_{1-n} \sum_{l=0}^{n-1} \left\{\nabla_{\mu} \Box^{(-l-1)} R \nabla^{\mu} \Box^{(-n+l)} R + 2\Box^{(-l-1)} R \Box^{(-n+l+1)} R\right\} = 0 \quad (17)$$

Our problem can be defined as following: To find the function for the scale factor a(t) in terms of the parameter t. First of all, it will be useful, to look carefully at every term in equation 17 to understand what it is. From the derivation of the equations of motion we know that:

$$\bar{F}_1(\Box) = \sum_{n=1}^{\infty} f_{i-n} \Box^{-n} \quad (18)$$

with i=(1,2,3). Here i=1.

$$f_{i-n} = \bar{f}_{i-n} M^{2n} \quad (19)$$

With $\bar{f_{i-n}}$ constant, and M^{2n} infrared mass scale.

It will be of course significant for the solution of the problem, the following well known result in literature (Deser Woodard 2007) :

$$\frac{1}{\Box}R = -\int_0^t dt' \frac{1}{a^3(t')} \int_0^{t'} dt'' a^3(t'') R(t'') \quad (20)$$

Due to the intrinsic complexity of equation 20, and the presence of infinite summations, it is clearly extremely difficult, if not theoretically impossible to simply solve equation 17 for a(t). However it is possible to make an ansatz about $\frac{1}{\Box}R$, in order to find a solution for a(t). The anstaz can be verified once we know the solution of (eq 17), by plugging in the solution in equation 20. Of course it would be also extremely difficult to plug in the obtained solution for a(t) in equation 17, to see if it indeed verifies the equation, however it is conceivable that this might be feasible, in contrast with just solving equation 17 for a(t).

The first term is the classical Einstein-Hilbert term, this term has mass dimension 4. The second and third terms are due to the proposed corrections to the classical Einstein-Hilbert action. These terms are problematic to deal with because they both represent infinite sums. However, one would expect, in order for our equation to have a physical meaning, both of these sums, must give a specific finite value. From a dimensional point of view, we would expect the terms to have the same dimensions with the Hilbert term. We would therefore expect the second and third terms to have mass dimension 4.

There is a reason, of why the ansatz

$$\Box^{-1}R = wR + w_1 \tag{21}$$

is to be expected, with w and w_1 constants (here w has to have mass dimension -2, and w_1 must be dimensionless). This reason, has to do with infinite Sums. As we already mentioned we wouldn't expect the infinite sums to yield an infinite quantity, but a finite, quantity. In particular it is easy to see that any ansatz with $\Box^{-1}R \propto R^2$ or higher powers of R, would diverge if $M > 1$ in equation 19 and $R > 1$ in equation 2. But even if $M < 1$ or $R < 1$ it is impossible to calculate the converging summation without beforehand inserting a value for M and a(t).

The next step would be to plug in $\Box^{-1}R = wR + w_1$, in the trace equation (equation 17), so that we can calculate a(t).

We will start from inserting $\Box^{-1}R = wR + w_1$ in $6\Box\left\{\bar{F}_1(\Box)R\right\}$ of equation 17. From equation 18 we have :

$$6\Box\left\{\bar{F}_1(\Box)R\right\} = 6\Box\left\{\sum_{n=1}^{\infty} f_{1-n}\Box^{-n}R\right\} = \tag{22}$$

$$= 6\Box\left\{f_{1-1}\Box^{-1}R + f_{1-2}\Box^{-2}R + f_{1-3}\Box^{-3}R + ...\right\} \tag{23}$$

$$= 6\Box\left\{f_{1-1}\left\{wR + w_1\right\} + f_{1-2}\Box^{-1}\left\{wR + w_1\right\} + f_{1-3}\Box^{-1}\left\{\Box^{-1}\left\{wR + w_1\right\}\right\} + ...\right\} \tag{24}$$

We can use the property $\Box\Box^{-1}R = R$, which means that $\Box[wR + w_1] = R$ or$\Box[wR] = R$. (Since $\Box w_1 = 0$, [this can be seen from the definition of the operator $\Box = -\frac{1}{a^3}\frac{d}{dt}(a^3\frac{d}{dt})$, acting on a constant, the result would always be zero.]) In addition, we have to use the ansatz $\Box^{-1}w_1 = w_2$. (Since we need to use equation 20 to calculate the result, but we need to know what is a(t) in order to perform the integrals, we have no other choice but use an ansatz). Where w_2 is a new constant with mass dimension -2.

$$= 6\Box\left\{f_{1-1}\left\{wR + w_1\right\} + f_{1-2}\left\{w(wR + w_1) + w_2\right\} + f_{1-3}\Box^{-1}\left\{w(wR + w_1) + w_2\right\} + ...\right\} \tag{25}$$

The $\Box$ acting on constants w_1, w_2, w_3, ... will make them zero. Hence we are left with:

$$= 6\Box\left\{f_{1-1}\left\{wR\right\} + f_{1-2}\left\{w^2R\right\} + f_{1-3}\left\{w^3R\right\} + ... + f_{1-n}\left\{w^nR\right\}\right\} \tag{26}$$

$$= hR \tag{27}$$

Where h, is a new constant, with mass dimension 2 .

Next we calculate the double sum of equation 17, with the aid of our ansatz :

$$= \sum_{n=1}^{\infty} f_{1-n}\sum_{l=0}^{n-1}\left\{\nabla_\mu\Box^{(-l-1)}R\nabla^\mu\Box^{(-n+l)}R + 2\Box^{(-l-1)}R\Box^{(-n+l+1)}R\right\} \tag{28}$$

We note that $\Box^{(-n+l+1)}R$ will yield R under the double summation, since for l=0 , n=1 we get R, for l=1 n=2 we get again R etc. We can also see that $\Box^{(-n+l)}$ will become $\Box^{-1}$ under the same argument.

$$= \sum_{n=1}^{\infty} f_{1-n}\sum_{l=0}^{n-1}\left\{\nabla_\mu\Box^{-l}\Box^{-1}R\nabla^\mu\Box^{(-1)}R + 2\Box^{-l}\Box^{-1}R(R)\right\} \tag{29}$$

$$= \sum_{n=1}^{\infty} f_{1-n}\sum_{l=0}^{n-1}\left\{\nabla_\mu\Box^{-l}(wR + w_1)\nabla^\mu(wR + w_1) + 2\Box^{-l}(wR + w_1)(R)\right\} \tag{30}$$

In order to calculate the above summation we note that:

$$\Box^{-n}R = gR + g_1 \tag{31}$$

with n being any integer, positive real number from 1 to infinity i.e we obtain the same result up to constants, as $\Box^{-1}R = wR + w_1$

Hence the Sum (30) can be written as:

$$= \nabla_\mu(gR + g_1)\nabla^\mu(wR + w_1) + 2(gR + g_1)(R) \tag{32}$$

$$= [g\theta_t(R)][w\theta^t(R)] + 2(gR + g_1)(R) \tag{33}$$

here g is dimensionless (because it has been multiplied with f_{1-n}), and w has mass dimension -2. The above equation can be written as:

$$= [c_3(R')^2] + qR^2 + vR \tag{34}$$

where we have used some new unkown constants c_3, q and v.

c_3 has mass dimension -2, and q has mass dimension 0, and v has mass dimension 2.

Hence, looking at the above result as well as the result (27) we can write equation 17 as:

$$c_1R + c_2R^2 + c_3(R')^2 = 0 \tag{35}$$

Our ansatz is just an assumption, which needs to be verified. If we plug in the solution of the above equation in equation 20, and we find the relastionship: $\Box^{-1}R = wR + w_1$, then all the steps from equation 17 up to the solution of equation 35 would be mathematically justified.

SOLUTIONS OF EQUATION 35

One obvious solution of equation 35 is R=0, which from equation 2 would mean:

$$R = 6\left[\frac{\ddot{a}(t)}{a(t)} + \left(\frac{\dot{a}(t)}{a(t)}\right)^2\right] = 0 \tag{36}$$

which has solution

$$a(t) \propto t^{1/2} \tag{37}$$

A second solution of equation 35, would be the Ricci scalar to be constant, in which case $R' = 0$. In this case equation 35 becomes:

$$c_1R + c_2R^2 = 0 \tag{38}$$

$$\{c_1 + c_2R\}\, R = 0 \tag{39}$$

$$c_1 + c_2R = 0 \tag{40}$$

this equation gives the following solution after using equation 2 :

$$a(t) = a_0 e^{bt} \tag{41}$$

where a_0 is a dimensionless constant, defined so that $a = a_0$ at t=0, and b is a constant with mass dimension 1, so that e^{bt} is dimensionless. The scale factor a(t) itself is dimensionless as is the metric tensor $g_{\mu\nu}$.

Next we calculate the Ricci scalar for $a = a_0 e^{bt}$ using equation 2. We find it to be:

$$R = -6\left\{\frac{a_0 b^2 e^{bt}}{a_0 e^{bt}} + \left\{\frac{a_0 b e^{bt}}{a_0 e^{bt}}\right\}^2\right\} \tag{42}$$

$$R = -6\left\{b^2 + b^2\right\} \tag{43}$$

$$R = -12b^2 \tag{44}$$

Substituting equation 44 in equation 40, we obtain:

$$c_1 - c_2(12b^2) = 0 \tag{45}$$

$$c_1 = c_2(12b^2) \tag{46}$$

$$b = \sqrt{\left\{\frac{c_1}{12c_2}\right\}} \tag{47}$$

The solution Eq 37 it is obvious that satisfies the trace equation eq 17, since R=0 in this case. Next we see that our solution eq 41 **does not** satisfy the ansatz $\Box^{-1}R = wR + w_1$. Equation 20 gives:

$$\Box^{-1}R = \frac{1}{\Box}R = -\int_0^t dt' \frac{1}{a^3(t')}\int_0^{t'} dt'' a^3(t'')R(t'') \tag{48}$$

$$\Box^{-1}R = \frac{1}{\Box}R = -\int_0^t dt' \frac{1}{{a_0}^3 e^{3bt}}\int_0^{t'} dt'' {a_0}^3 e^{3bt}\frac{(-12b^2)}{3b} \tag{49}$$

$$\Box^{-1}R = -\frac{4}{3}(3bt + e^{-3bt} - 1) \tag{50}$$

Check: We can use the identity$\Box\Box^{-1}R = R$ to check the validity of above significant result.

$$\Box(\Box^{-1}R) = R \tag{51}$$

$$\Box(\Box^{-1}R) = R \tag{52}$$

$\Box$ in an FRW background is defined by: $\Box = -(\frac{d^2}{dt^2} + 3H\frac{d}{dt})$.

$$\Box\left\{-\frac{4}{3}(3bt + e^{-3bt} - 1)\right\} = -((\Box^{-1}R)^{''} + 3H(\Box^{-1}R)^{'}) = 12b^2 = R \tag{53}$$

Therefore we must use a different ansatz to solve equation 17. This ansatz will be:

$$\Box^{-1}R = -\frac{4}{3}(3bt + e^{-3bt} - 1)) \tag{54}$$

with b being dimensionless constant.
Eq 17 can be written as:

$$-(M_p)^2 R + (c_1)R\{f(t)\} + (c_2)(R')^2\{f(t)\} + c_3 R^2\{f(t)\} = 0 \tag{55}$$

$$-(M_p)^2 R + f(t)[(c_1)R + (c_2)(R')^2 + (c_3)(R)^2] = 0 \tag{56}$$

we now plug in the solution : $a = a_0 e^{bt}$, in the equation above. This will yield:

$$-(M_p)^2 R + f(t)[(c_1)R + (c_3)(R)^2] = 0 \tag{57}$$

$$-(M_p)^2 R + f(t)[(c_1) + (c_3)(R)]R = 0 \tag{58}$$

$$\left\{-(M_p)^2 + f(t)[(c_1) + (c_3)(R)]\right\} R = 0 \tag{59}$$

$$\left\{-(M_p)^2 + f(t)[(c_1) + (c_3)(R)]\right\} = 0 \tag{60}$$

$$\left\{-(M_p)^2 + f(t)[(c_1) + (c_3)(R)]\right\} = 0 \tag{61}$$

$$(c_1) + (c_3)(R) = \frac{(M_p)^2}{f(t)} \tag{62}$$

as we will see later $f(t) \to \infty$

$$(c_1) + (c_3)(R) = 0 \tag{63}$$

hence we see that the solution $a = a_0 e^{bt}$, verifies the above equation.